Systems and Formalisms Laboratory

A Puzzle: Data Structures with Holes

Yawen Guan, Clément Pit-Claudel

EPFL

Datatypes

$$D \triangleq$$
$$0 \mid 1 \mid x \mid D_1 + D_2 \mid D_1 \cdot D_2 \mid \mu x.\, D$$

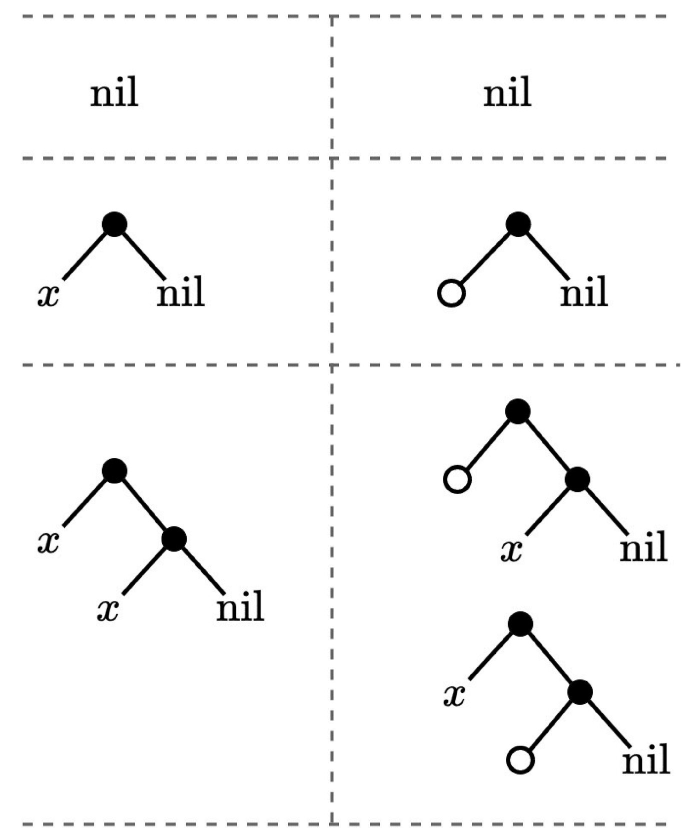
--------- Examples ---------

$$\text{bool} \triangleq 1 + 1$$
$$\text{option } x \triangleq x + 1$$
$$\text{nat} \triangleq \mu x.\, 1 + x$$
$$\text{list } x \triangleq \mu y.\, 1 + x \cdot y$$
$$\text{btree } x \triangleq \mu y.\, 1 + x \cdot y^2$$
$$\text{ttree } x \triangleq \mu y.\, 1 + x \cdot y^3$$
$$\text{ftree} \triangleq \mu x.\, \mu y.\, 1 + x \cdot y$$

One-hole Context

[McBride 2001]

A one-hole context c for element x within data structure t is a structure representing t with a hole replacing x.



the type of list's one-hole contexts:

$$\text{list}'\ x = \partial_x(\text{list } x) = (\text{list } x)^2$$

Rules

[McBride 2001]

$$\begin{aligned}
\partial_x x &\mapsto 1 \\
\partial_x y &\mapsto 0 \quad (y \neq x) \\
\partial_x 0 &\mapsto 0 \\
\partial_x 1 &\mapsto 0 \\
\partial_x (D_1 + D_2) &\mapsto \partial_x D_1 + \partial_x D_2 \\
\partial_x (D_1 \cdot D_2) &\mapsto \partial_x D_1 \cdot D_2 + D_1 \cdot \partial_x D_2 \\
\partial_x (\mu y.\, F) &\mapsto \mu z.\, \underline{(\partial_x F)|_{y=\mu y.\, F}}_z + \\
&\qquad \underline{(\partial_y F)|_{y=\mu y.\, F}}_z \cdot z \\
\partial_x (F|_{y=S}) &\mapsto (\partial_x F)|_{y=S} + (\partial_y F)|_{y=S} \cdot \partial_x S \\
\partial_x \underline{T}_x &\mapsto 0 \\
\partial_x \underline{T}_y &\mapsto \underline{\partial_x T}_y \quad (y \neq x)
\end{aligned}$$

Notation: $\underline{T}_x$ means x is not in T.

Partial differentiation!

Two perspectives of any-hole contexts

substitution perspective

$$T(x+1)$$

option x (x or hole)

The any-hole contexts of a data structure are structures where each element may be replaced by a hole.

$$\overset{?\ ?}{?\!=\!?}$$
$$?\ ?\ ?$$

derivative perspective

$$\sum_{n \geq 0} \frac{\partial_x^n T(x)}{n!}$$

nth derivative (n-hole context)

permutation of holes

any number of holes

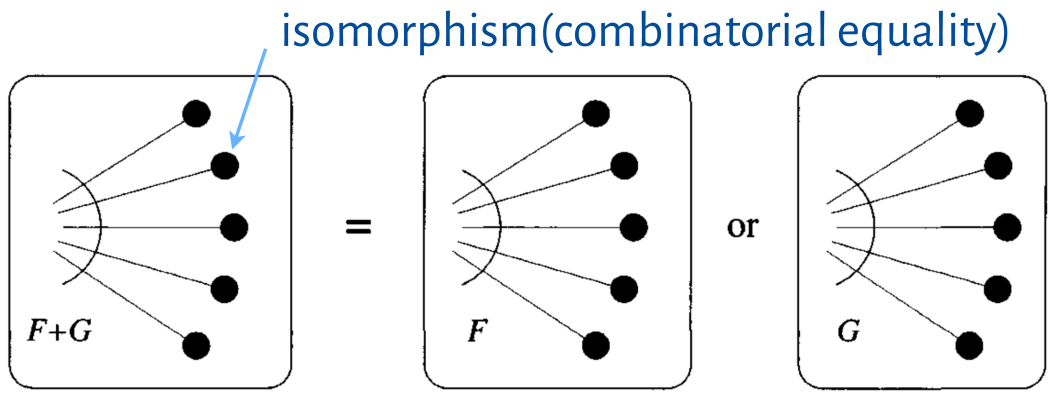
The any-hole contexts of a data structure are the union of its n-hole contexts modulo permutation.

Combinatorial Species Theory

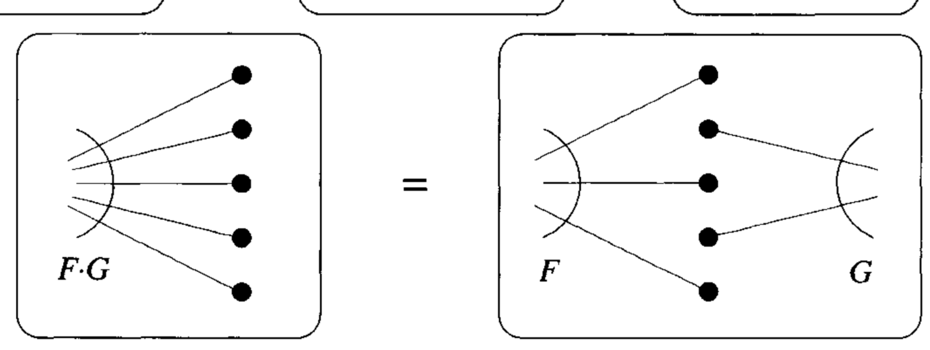
[Joyal 1986; Bergeron et al. 1997]

nice operation on species ~ nice operation on generating series

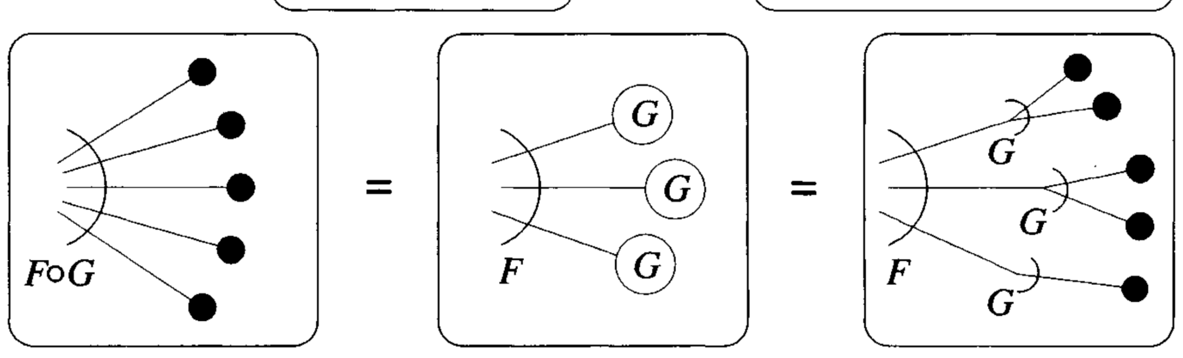
isomorphism(combinatorial equality)



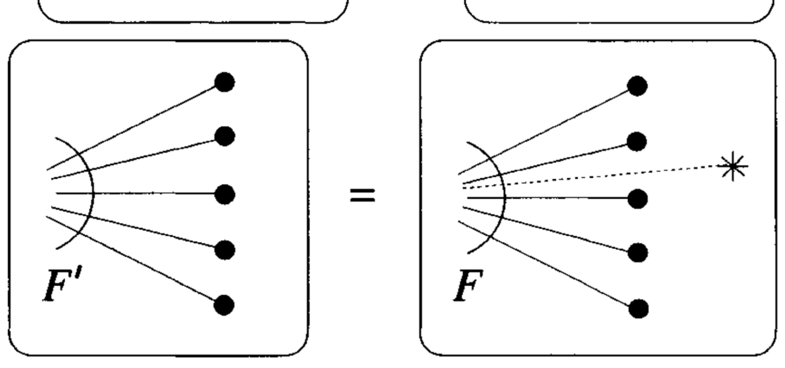
$$\sim \quad (F+G)(x) = F(x) + G(x)$$



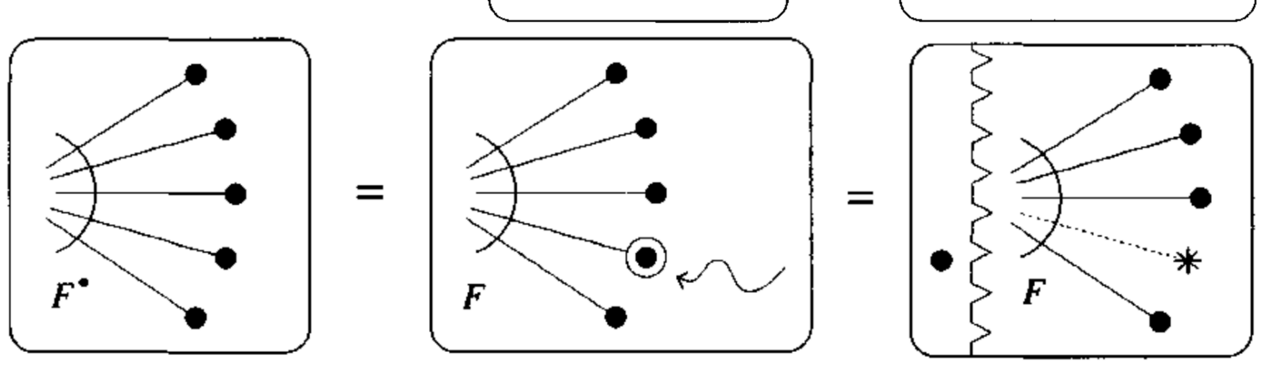
$$\sim \quad (F \cdot G)(x) = F(x)G(x)$$



$$\sim \quad (F \circ G)(x) = F(G(x))$$

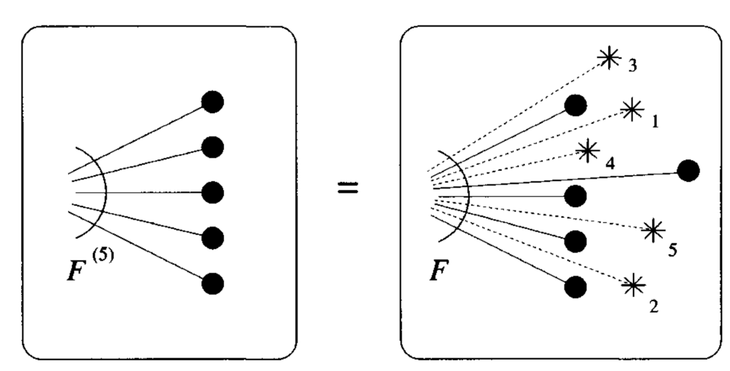


$$\sim \quad F'(x) = \frac{d}{dx} F(x)$$

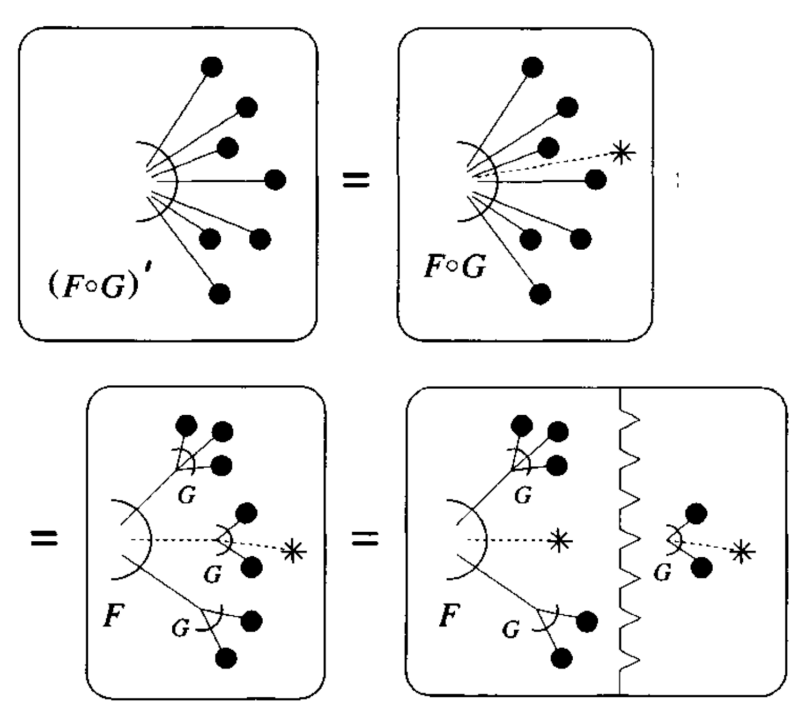


$$\sim \quad F^{\bullet}(x) = xF'(x) = x\frac{d}{dx}F(x)$$

Distinctness



Chain Rule



$$(F \circ G)' = (F' \circ G) \cdot G'$$